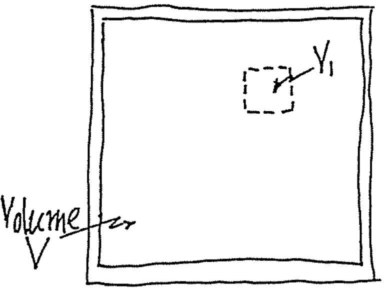
Class Work: Spatial Distribution of Gas Molecules

N molecules in volume V (e.g. $\sim 10^{22}$ in a litre or $\sim 10^{25}$ in m^3)



V_1 = a smaller volume inside V

System is in equilibrium.

Q: How many molecules are there in V_1?
What is the fluctuation in this number?

- Expectations from common sense:

(1) At equilibrium, density of gas is uniform in V

$$\Rightarrow \text{\# molecules in } V_1 = \left(\frac{V_1}{V}\right)\cdot N$$

(2) As molecules are constantly moving, this number fluctuates.

Q: How could we get at these common sense expectations mathematically, and more?

The power of ignorance!

- We know that a molecule must be somewhere within V.
- Simplest assumption: a molecule is equally likely to be anywhere in V

p = probability that a molecule is in $V_1 = \frac{V_1}{V}$

$q = 1 - p$ = probability that a molecule is not in V_1

Key Results [Details worked out in class]

$\mathcal{P}_N(n)$ = Prob. having exactly n molecules in V_1 (out of N)

$$= \frac{N!}{n!(N-n)!} p^n q^{N-n} \quad \text{(binomial distribution)}$$

Q1: Mean number in V_1?

$$\langle n \rangle = \sum_{n=0}^{N} n \mathcal{P}_N(n) = \overset{\text{steps}}{\cdots} = Np \quad \text{(as expected)}$$

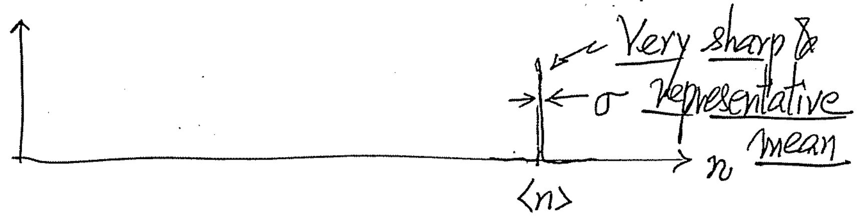
Q2: Variance? (Fluctuations)

$$\sigma^2 \equiv \langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 = \langle n(n-1) \rangle + \langle n \rangle - \langle n \rangle^2$$

$$= Npq$$

σ = standard deviation $= \sqrt{Npq} \sim \sqrt{N}$

Relative fluctuation $= \frac{\sigma}{\langle n \rangle} \sim \frac{1}{\sqrt{N}}$ (tiny tiny as $N \to \infty$)



P.T.O.

One-dimensional random walk: p = prob. stepping to the right
q = prob. stepping to the left

For a long walk with $N \gg 1$ steps, what is the prob. $P_N(m)$ that the walker is m steps to the right of the origin (m could be +ve or −ve)? Mean displacement? Standard deviation?

Q3 : <u>Where does $P_N(n)$ peak?</u>

Same as where $\underbrace{\ln P_N(n)}_{\text{easier to handle}}$ peaks

- Take $\ln P_N(n)$
- Apply Stirling's formula
- Take $\frac{d}{dn} \ln P_N(n)$
- $\bar{n}$ determined by $\left.\frac{d}{dn} \ln P_N(n)\right|_{n=\bar{n}} = 0$
- Found $\bar{n} = Np = \langle n \rangle$

$\therefore$ $P_N(n)$ peaks at the mean value.

Q4 : <u>What is the functional form of $P_N(n)$ near the peak?</u>

- Use Taylor expansion around $\bar{n}$

i.e. for $n \approx \bar{n}$

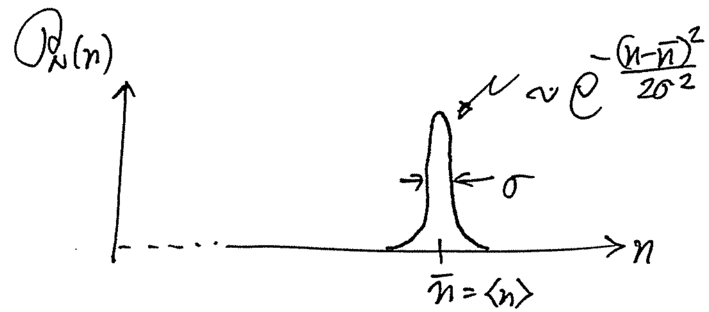
$$\ln P_N(n) \approx \ln P_N(\bar{n}) + \underbrace{\cancel{\left.\frac{d}{dn} \ln P_N(n)\right|_{n=\bar{n}}}}_{0} \cdot (n-\bar{n}) + \underbrace{\frac{1}{2}\left.\frac{d^2}{dn^2} \ln P_N(n)\right|_{n=\bar{n}} \cdot (n-\bar{n})^2}_{-\frac{(n-\bar{n})^2}{2\sigma^2}}$$

where $\sigma^2 = Npq$

$\therefore$ Near $n = \bar{n}$,

$$P_N(n) \approx P_N(\bar{n}) \underbrace{e^{-\frac{(n-\bar{n})^2}{2\sigma^2}}}_{\text{called Gaussian distribution}}$$

Blow up picture of $P_N(n)$ near $\bar{n}$:



- See appendices on gaussian distribution and central limit theorem

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<u>Remarks</u>: Things to learn from the example

- Formal ways of evaluating mean and variance given a distribution
- Vast number of entities (particles) gives sharp & representative Mean
- Vast number of independent entities (particles) gives Gaussian (Normal) distribution

<u>Extensions</u>

- Gaussian distribution is completely characterized by mean & variance
- Central Limit Theorem.